Mathematical Methods

2018 Subject Outline

Stage 2

For teaching

- in Australian and SACE International schools from January 2018 to December 2018
- in SACE International schools only, from May/June 2018 to March 2019

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INTRODUCTION

SUBJECT DESCRIPTION

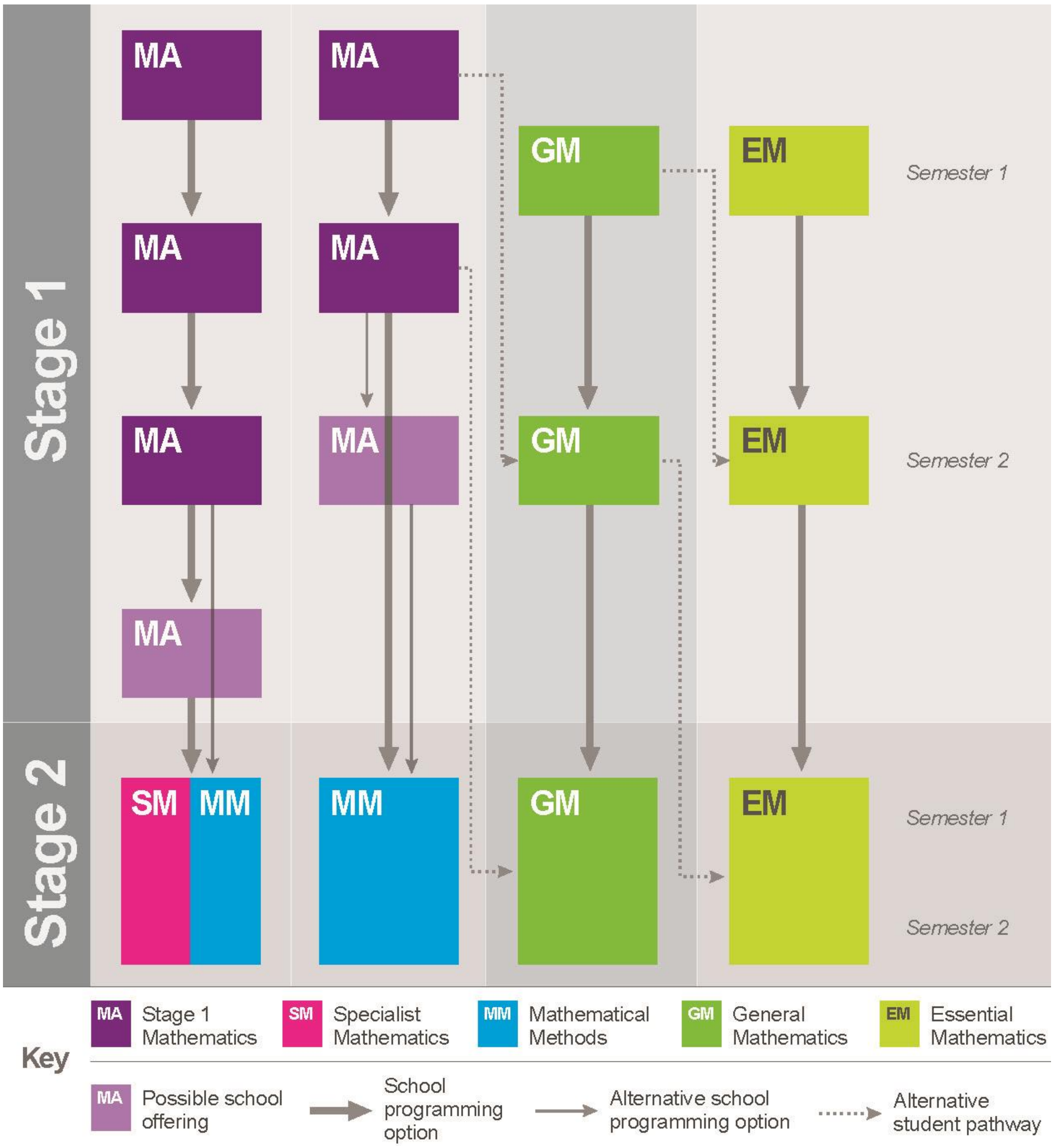
Mathematical Methods is a 20-credit subject at Stage 2.

Mathematical Methods develops an increasingly complex and sophisticated understanding of calculus and statistics. By using functions and their derivatives and integrals, and by mathematically modelling physical processes, students develop a deep understanding of the physical world through a sound knowledge of relationships involving rates of change. Students use statistics to describe and analyse phenomena that involve uncertainty and variation.

Mathematical Methods provides the foundation for further study in mathematics, economics, computer sciences, and the sciences. It prepares students for courses and careers that may involve the use of statistics, such as health or social sciences. When studied together with Specialist Mathematics, this subject can be a pathway to engineering, physical science, and laser physics.

MATHEMATICAL OPTIONS

The diagram below represents the possible mathematical options that students might study at Stage 1 and Stage 2.



Notes:

Although it is advantageous for students to study Australian Curriculum 10 and 10A in Year 10, the 10A curriculum per se is not a prerequisite for the study of Specialist Mathematics and Mathematical Methods. The essential aspects of 10A are included in the curriculum for Specialist Mathematics and Mathematical Methods.

Mathematical Methods can be studied as a single subject; however, Specialist Mathematics is designed to be studied together with Mathematical Methods.

CAPABILITIES

The capabilities connect student learning within and across subjects in a range of contexts. They include essential knowledge and skills that enable people to act in effective and successful ways.

The SACE identifies seven capabilities. They are:

- literacy
- numeracy
- information and communication technology (ICT) capability
- critical and creative thinking
- personal and social capability
- ethical understanding
- intercultural understanding.

Literacy

In this subject students develop their literacy capability by, for example:

- communicating mathematical reasoning and ideas for different purposes, using appropriate language and representations, such as symbols, equations, tables, and graphs
- interpreting and responding to appropriate mathematical language and representations
- analysing information and explaining mathematical results.

Mathematics provides a specialised language to describe and analyse phenomena. It provides a rich context for students to extend their ability to read, write, visualise, and talk about situations that involve investigating and solving problems.

Students apply and extend their literacy skills and strategies by using verbal, graphical, numerical, and symbolic forms of representing problems and displaying statistical information. Students learn to communicate their findings in different ways, using different systems of representation.

Numeracy

Being numerate is essential for participating in contemporary society. Students need to reason, calculate, and communicate to solve problems. Through the study of mathematics, they understand and use mathematical skills, concepts, and technologies in a range of contexts that can be applied to:

- using measurement in the physical world
- gathering, representing, interpreting, and analysing data
- using spatial sense and geometric reasoning
- investigating chance processes
- using number, number patterns, and relationships between numbers
- working with graphical, statistical and algebraic representations, and other mathematical models.

Information and communication technology (ICT) capability

In this subject students develop their information and communication technology capability by, for example:

- understanding the role of electronic technology in the study of mathematics
- making informed decisions about the use of electronic technology
- understanding the mathematics involved in computations carried out using technologies, so that reasonable interpretations can be made of the results.

Students extend their skills in using technology effectively and processing large amounts of quantitative information.

Students use ICT to extend their theoretical mathematical understanding and apply mathematical knowledge to a range of problems. They use software relevant for study and/or workplace contexts. This may include tools for statistical analysis, algorithm generation, data representation and manipulation, and complex calculation. They use digital tools to make connections between mathematical theory, practice, and application; for example, to use data, address problems, and operate systems in particular situations.

Critical and creative thinking

In this subject students develop critical and creative thinking by, for example:

- building confidence in applying knowledge and problem-solving skills in a range of mathematical contexts
- developing mathematical reasoning skills to think logically and make sense of the world
- understanding how to make and test projections from mathematical models
- interpreting results and drawing appropriate conclusions
- reflecting on the effectiveness of mathematical models, including the recognition of assumptions, strengths, and limitations
- using mathematics to solve practical problems and as a tool for learning
- making connections between concrete, pictorial, symbolic, verbal, written, and mental representations of mathematical ideas
- thinking abstractly, making and testing conjectures, and explaining processes.

Problem-solving in mathematics builds students’ depth of conceptual understanding and supports development of critical and creative thinking. Learning through problem-solving helps students when they encounter new situations. They develop their creative and critical thinking capability by listening, discussing, conjecturing, and testing different strategies. They learn the importance of self-correction in building their conceptual understanding and mathematical skills.

Personal and social capability

In this subject students develop their personal and social capability by, for example:

- arriving at a sense of self as a capable and confident user of mathematics through expressing and presenting ideas in a variety of ways
- appreciating the usefulness of mathematical skills for life and career opportunities and achievements
- understanding the contribution of mathematics and mathematicians to society.

The elements of personal and social competence relevant to mathematics include the application of mathematical skills for informed decision-making, active citizenship, and effective self-management. Students build their personal and social competence in mathematics through setting and monitoring personal and academic goals, taking initiative, and building adaptability, communication, and teamwork.

Students use mathematics as a tool to solve problems they encounter in their personal and working lives. They acquire a repertoire of strategies and build the confidence needed to:

- meet the challenges and innovations of a rapidly changing world
- be the designers and innovators of the future, and leaders in their fields.

Ethical understanding

In this subject students develop their ethical understanding by, for example:

- gaining knowledge and understanding of ways in which mathematics can be used to support an argument or point of view
- examining critically ways in which the media present particular perspectives
- sharing their learning and valuing the skills of others
- considering the social consequences of making decisions based on mathematical results
- acknowledging and learning from errors rather than denying findings and/or evidence.

Areas of ethical understanding relevant to mathematics include issues associated with ethical decision-making and working collaboratively as part of students' mathematically related explorations. They develop ethical understanding in mathematics through considering social responsibility in ethical dilemmas that may arise when solving problems in personal, social, community, and/or workplace contexts.

Intercultural understanding

In this subject students develop their intercultural understanding by, for example:

- understanding mathematics as a body of knowledge that uses universal symbols which have their origins in many cultures
- understanding how mathematics assists individuals, groups, and societies to operate successfully across cultures in the global, knowledge-based economy.

Mathematics is a shared language that crosses borders and cultures, and is understood and used globally.

Students read about, represent, view, listen to, and discuss mathematical ideas. They become aware of the historical threads from different cultures that have led to the current bodies of mathematical knowledge. These opportunities allow students to create links between their own language and ideas and the formal language and symbols of mathematics.

ABORIGINAL AND TORRES STRAIT ISLANDER KNOWLEDGE, CULTURES, AND PERSPECTIVES

In partnership with Aboriginal and Torres Strait Islander communities, and schools and school sectors, the SACE Board of South Australia supports the development of high-quality learning and assessment design that respects the diverse knowledge, cultures, and perspectives of Indigenous Australians.

The SACE Board encourages teachers to include Aboriginal and Torres Strait Islander knowledge and perspectives in the design, delivery, and assessment of teaching and learning programs by:

- providing opportunities in SACE subjects for students to learn about Aboriginal and Torres Strait Islander histories, cultures, and contemporary experiences
- recognising and respecting the significant contribution of Aboriginal and Torres Strait Islander peoples to Australian society
- drawing students' attention to the value of Aboriginal and Torres Strait Islander knowledge and perspectives from the past and the present
- promoting the use of culturally appropriate protocols when engaging with and learning from Aboriginal and Torres Strait Islander peoples and communities.

SACE NUMERACY REQUIREMENT

Completion of 10 or 20 credits of Stage 1 Mathematics with a C grade or better, or 20 credits of Stage 2 Mathematical Methods with a C– grade or better, will meet the numeracy requirement of the SACE.

LEARNING SCOPE AND REQUIREMENTS

LEARNING REQUIREMENTS

The learning requirements summarise the key skills, knowledge and understanding that students are expected to develop and demonstrate through learning in Stage 2 Mathematical Methods.

In this subject, students are expected to:

1. understand mathematical concepts, demonstrate mathematical skills, and apply mathematical techniques
2. investigate and analyse mathematical information in a variety of contexts
3. think mathematically by posing questions, solving problems, applying models, and making, testing, and proving conjectures
4. interpret results, draw conclusions, and determine the reasonableness of solutions in context
5. make discerning use of electronic technology to solve problems and to refine and extend mathematical knowledge
6. communicate mathematically and present mathematical information in a variety of ways.

CONTENT

Stage 2 Mathematical Methods is a 20-credit subject.

Stage 2 Mathematical Methods focuses on the development of mathematical skills and techniques that enable students to explore, describe, and explain aspects of the world around them in a mathematical way. It places mathematics in relevant contexts and deals with relevant phenomena from the students' common experiences, as well as from scientific, professional, and social contexts.

The coherence of the subject comes from its focus on the use of mathematics to model practical situations, and on its usefulness in such situations. Modelling, which links the two mathematical areas to be studied, calculus and statistics, is made more practicable by the use of electronic technology.

The ability to solve problems based on a range of applications is a vital part of mathematics in this subject. As both calculus and statistics are widely applicable as models of the world around us, there is ample opportunity for problem-solving throughout this subject.

Stage 2 Mathematical Methods consists of the following six topics:

- Topic 1: Further differentiation and applications
- Topic 2: Discrete random variables
- Topic 3: Integral calculus
- Topic 4: Logarithmic functions
- Topic 5: Continuous random variables and the normal distribution
- Topic 6: Sampling and confidence intervals.

The suggested order of the topics is a guide only; however, students study all six topics. If Mathematical Methods is to be studied in conjunction with Specialist Mathematics, consideration should be given to appropriate sequencing of the topics across the two subjects.

Each topic consists of a number of subtopics. These are presented in the subject outline in two columns as a series of key questions and key concepts, side by side with considerations for developing teaching and learning strategies.

The key questions and key concepts cover the prescribed areas for teaching, learning, and assessment in this subject. The considerations for developing teaching and learning strategies are provided as a guide only.

A problem-based approach is integral to the development of the mathematical models and associated key concepts in each topic. Through key questions teachers can develop the key concepts and processes that relate to the mathematical models required to address the problems posed.

The considerations for developing teaching and learning strategies present problems for consideration and guidelines for sequencing the development of the concepts. They also give an indication of the depth of treatment and emphases required.

Although the material for the external examination will be based on the key questions and key concepts outlined in the six topics, the considerations for developing teaching and learning strategies may provide useful contexts for examination questions.

Students should have access to technology, where appropriate, to support the computational aspects of these topics.

Calculus

The following three topics relate to the study of calculus:

- Topic 1: Further differentiation and applications
- Topic 3: Integral calculus
- Topic 4: Logarithmic functions

Calculus is essential for developing an understanding of the physical world, as many of the laws of science are relationships involving rates of change. The study of calculus provides a basis for understanding rates of change in the physical world, and includes the use of functions, and their derivatives and integrals, in modelling physical processes.

In this area of study, students gain a conceptual grasp of calculus, and the ability to use its techniques in applications. This is achieved by working with various kinds of mathematical models in different situations, which provide a context for investigating and analysing the mathematical function behind the mathematical model.

The study of calculus continues from Stage 1 Mathematics with the derivatives of exponential, logarithmic, and trigonometric functions and their applications, together with differentiation techniques and applications to optimisation problems and graph sketching. It concludes with integration, both as a process that reverses differentiation and as a way of calculating areas. The fundamental theorem of calculus as a link between differentiation and integration is emphasised.

Statistics

The following three topics relate to the study of statistics:

- Topic 2: Discrete random variables
- Topic 5: Continuous random variables and the normal distribution
- Topic 6: Sampling and confidence intervals

The study of statistics enables students to describe and analyse phenomena that involve uncertainty and variation. In this area of study, students move from asking statistically sound questions towards a basic understanding of how and why statistical decisions are made. The area of study provides students with opportunities and techniques to examine argument and conjecture from a statistical point of view. This involves working with discrete and continuous variables, and the normal distribution in a variety of contexts; learning about sampling in certain situations; and understanding the importance of sampling in statistical decision-making.

Topic 1: Further differentiation and applications

Subtopic 1.1: Introductory differential calculus

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is the derivative of x^n, where n is a rational number? Establish the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$ from first principles when $y = x^n$, where n is a rational number Determine the derivative of linear combinations of power functions involving rational exponents	Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to differentiate functions such as $x^{\frac{1}{2}}$.
What types of problems can be solved by finding the derivatives of functions? • The derivative of a function can be used to find the slope of tangents to the function, and hence the equation of the tangent and/or normal at any point on the function	This is an extension of Topic 6: Introduction to differential calculus from Stage 1 Mathematics.
• When an object's displacement is described by a function, the derivative can be used to find the instantaneous velocity • The sign diagram of the derivative function can be used to find when the function is increasing, decreasing, and stationary • The derivative of a function can be used to determine the rate of change, and the position of any local maxima or minima	Polynomials and linear combinations of power functions can be used to model many scenarios, such as areas of quadrilaterals, volumes of boxes, numbers of people at an event, and the speed of objects.
	These applications of calculus are included here so that conceptual understanding is built with functions that students have differentiated mainly in Stage 1. As further differentiation skills are developed, these applications are revisited.

Subtopic 1.2: Differentiation rules

Key questions and key concepts	Considerations for developing teaching and learning strategies
What are the different algebraic structures of functions? • Functions can be classified as sums, products, or quotients of simpler functions through an analysis of the use of grouping symbols (brackets) and a knowledge of the order of operations	Correct identification of the algebraic structure of a function allows the correct differentiation process to be applied.
How is it possible to differentiate composite functions of the form $h(x)=f(g(x))$, with at most one application of the chain rule? • $h(x)=f(g(x))$ has a derivative $h'(x)=f'(g(x))g'(x)$ This is called the chain rule	The chain rule can be verified for a number of simple examples by establishing: • numerical results for the function's derivative, using electronic technology • the same results algebraically by evaluating the product of the two factors proposed by the chain rule. An algebraic approach would be to compare applying the chain rule and simplifying (simplifying and differentiating term by term).
How do we differentiate the product of two functions $h(x)=f(x)g(x)$? • $h(x)=f(x)g(x)$ has a derivative $h'(x)=f'(x)g(x)+f(x)g'(x)$ This is called the product rule	Methods similar to those used to introduce the chain rule can be used for the product rule. A more general justification follows, allowing progression to using the rule when the alternative expansions are too cumbersome or not available.
How do we differentiate functions of the form $h(x)=\frac{f(x)}{g(x)}$? • $h(x)=\frac{f(x)}{g(x)}$ has a derivative $h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$ This is called the quotient rule	The quotient rule can be established by applying the product rule to $f(x)=h(x)\left(g(x)\right)^{-1}$

Subtopic 1.3: Exponential functions

Key questions and key concepts	Considerations for developing teaching and learning strategies
What form does the derivative function of an exponential function $y = ab^x$ take? • The derivative of $y = ab^x$ is a multiple of the original function	Working in the context of a population with a simple doubling rule $P(t) = 2^t$ and using electronic technology, it is possible to show that the derivative seems to be a multiple of the original growth function. This leads to a number of 'What if …' questions: • What if the derivative is investigated numerically by examining $\dfrac{2^{t+h} - 2^t}{h} = \dfrac{2^h - 1}{h}.2^t$ for smaller and smaller values of h? • What if the function considered is $P(t) = 5^t$? • What if a base could be found that provided for $P'(t) = P(t)$?
For what value of b is the derivative equal to the original function $y = b^x$? • There exists an irrational number e so that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y = e^x$ The approximate value of e is 2.7182818	Electronic technology can be used to graph both the original function and the derivative on the same axes. Changing the value of b from 2 to 3 to 2.8 to 2.7 to 2.72, can lead to the conjecture that there is a base close to 2.72 for which the original function and the derivative are the same function. Numerical and graphical investigations of the expression $\left(1 + \dfrac{1}{t}\right)^t$ also lead to the existence of the number conventionally called e.
What are the derivatives of $y = e^x$ and $y = e^{f(x)}$? • The chain rule can be used to show that the derivative of $y = e^{f(x)}$ is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{f(x)} f'(x)$	The chain rule, product rule, and quotient rule are considered with the inclusion of e^x and $e^{f(x)}$.

Key questions and key concepts	Considerations for developing teaching and learning strategies
What do the graphs of exponential functions look like? • Many exponential functions show growth or decay. This growth/decay may be unlimited or asymptotic to specific values	Electronic technology can be used to explore the different features of various types of exponential graphs. Suitable functions for exploration include the surge function $f(x) = axe^{-bx}$ and the logistic function $P(t) = \frac{L}{1+Ae^{-bt}}$. Exploration of the location of the turning point of $y = \frac{x+a}{e^x}$ can provide an opportunity for the development, testing, and proof of conjectures.
What types of problems can be solved by finding the derivatives of exponential functions? Derivatives can be used to: • find the slope of tangents to the graphs of exponential functions, and hence the equation of the tangent and/or normal at any point on the function • find the instantaneous velocity, when an object's displacement is described by an exponential function • determine the rate of change, the position of any local maxima or minima, and when the function is increasing or decreasing	These applications of calculus were developed in Topic 6: Introduction to differential calculus, Stage 1 Mathematics, using polynomials and linear combinations of power functions. The use of e^x and $e^{f(x)}$ within the modelling function shows the need for a way to determine the exact value of *x* such that e^x is equal to specific values. This is covered in Topic 4: Logarithmic functions of this subject outline. For some examples, approximate values can be determined using electronic technology.
Use exponential functions and their derivatives to solve practical problems where exponential functions model actual examples	Exponential functions can be used to model many actual scenarios, including those involving growth and decay.

Subtopic 1.4: Trigonometric functions

Key questions and key concepts	Considerations for developing teaching and learning strategies
How can sine and cosine be considered as functions? • Graphing sine and cosine functions in Topic 3: Trigonometry, Stage 1 Mathematics introduced the concept of radian angle measure, the use of sine and cosine to define different aspects of the position of a moving point on a unit circle, and the ability to solve trigonometric equations • When t is a variable measured in radians (often time), $\sin t$ and $\cos t$ are periodic functions	Students reinforce their understanding of sine and cosine in terms of the circular motion model with an emphasis on the coordinates of the moving point as functions of time, t.
What are the derivatives of $y = \sin t$ and $y = \cos t$? • $y = \sin t$ has a derivative $\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$ • $y = \cos t$ has a derivative $\frac{\mathrm{d}y}{\mathrm{d}t} = -\sin t$	By inspecting the graphs of sine and cosine, and the gradients of their tangents, students conjecture that $(\cos t)' = -\sin t$ and $(\sin t)' = \cos t$. Numerical estimations of the limits and informal proofs based on geometric constructions are used. The derivative of $\sin t$ or $\cos t$ from first principles is not required but provides an extension of the first-principles process to calculate derivatives. A first-principles calculation from the limit definition provides a proof that the derivatives exist, and enables students to see the results in the context of the gradient-of-tangent interpretation of derivatives from which the limit definition is derived. This traditional calculation proceeds by first computing the limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$, which is derived graphically, numerically, and geometrically. The derivatives of the sine and cosine functions in general can be derived from this special case using the sums-to-product identities.

Key questions and key concepts	Considerations for developing teaching and learning strategies
How are the chain, product, and quotient rules applied with trigonometric functions? • The use of the quotient rule on $\frac{\sin t}{\cos t}$ allows the derivative of $\tan t$ to be found • Derivatives can be found for functions such as $x\sin x,\ \frac{e^x}{\cos x}$ • The chain rule can be applied to $\sin f(x)$ and $\cos f(x)$	This is an extension of differentiation rules, with a wider range of functions used to build the composite functions. The reciprocal trigonometric functions sec, cot and cosec, are not required in this subject.
What types of problems can be solved by finding the derivatives of trigonometric functions? Derivatives can be used to: • find the slope of tangents to the graphs of trigonometric functions, and hence the equation of the tangent and/or normal at any point on the function • find the instantaneous velocity, when an object's displacement is described by a trigonometric function • determine the rate of change, the position of any local maxima or minima, and when the function is increasing or decreasing	These applications of calculus were developed in Topic 6: Introduction to differential calculus, Stage 1 Mathematics, using polynomials and linear combinations of power functions, and using exponential functions.
Use trigonometric functions and their derivatives to solve practical problems where trigonometric functions model periodic phenomena	Trigonometric functions are used to model many periodic scenarios, such as tidal heights, temperature changes, and AC voltages.

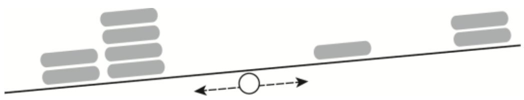
Subtopic 1.5: The second derivative

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is meant by the 'second derivative'? • The second derivative is the result of differentiating the derivative of a function • The notations y'', $f''(x)$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ can be used for the second derivative	Briefly consider the role of a derivative in describing the rate of change, to enable students to understand that the second derivative describes the rate of change of the first derivative. Graphical examples, where the rate of change is interpreted as the slope of the tangent, are used to show the relationships between the function, its derivative, and its second derivative.
What role does the second derivative play when studying motion along a straight line? • The second derivative of a displacement function describes the acceleration of a particle, and is used to determine when the velocity is increasing or decreasing	The use of acceleration functions to describe the changes in the velocity is a direct application of the second derivative's role in describing the rate of change of the first derivative.
How can the first and second derivative of a function be used to locate stationary points and points of inflection? • The sign diagram of the first and second derivative provides information to assist in sketching the graphs of functions • Stationary points occur when the first derivative is equal to zero, and may be local maxima, local minima, or stationary inflections • Points of inflection occur when the second derivative equals zero and changes sign, and may be classed as stationary or non-stationary • The second derivative can be used to describe the concavity of a curve • Whether the second derivative is positive, negative, or zero at a stationary point is used to determine the nature of the stationary point	Graphical analysis of functions, their derivatives, and second derivatives (using electronic technology) are used to explore the rules about turning points and points of inflection.

Topic 2: Discrete random variables

Subtopic 2.1: Discrete random variables

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is a random variable? • A random variable is a variable, the value of which is determined by a process, the outcome of which is open to chance. For each random variable, once the probability for each value is determined it remains constant	Use examples to introduce the concept of the random variable, such as the number of heads appearing on a coin tossed 10 times, the number of attempts taken to pass a driving test, and the length of a time before a phone loses its battery charge.
How are discrete random variables different from continuous random variables?	
• Continuous random variables may take any value (often within set limits)	Examples of continuous random variables include • height • mass.
• Discrete random variables may take only specific values	Examples of discrete random variables include the number of: • cars arriving at a set of traffic lights before it turns from red to green • phone calls made before a salesperson has sold 3 products • mutations on a strand of DNA • patients in a doctor's waiting room at any specific time • tropical cyclones annually, in a specific region.
What is a probability distribution of a discrete random variable and how can it be displayed? • A probability function specifies the probabilities for each possible value of a discrete random variable. This collection of probabilities is known as a probability distribution • A table or probability bar chart can show the different values and their associated probability • The sum of the probabilities must be 1	A suitable context to explore is the profit/loss obtained by a tradesperson who quotes \$1000 for a particular job. Three outcomes that each have a probability of occurring are that the: • quote is rejected because it was too high, resulting in zero profit • job is done, costing \$800 in materials and labour, resulting in a \$200 profit • job is done, costing \$1100 in materials and labour, resulting in a \$100 loss. It is appropriate to limit the scenarios to examples with variables that are a finite set of integers.

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is the difference between uniform discrete random variables and non-uniform discrete random variables? • For most discrete random variables the probabilities for the different outcomes are different, whereas uniform discrete random variables have the same probability for each outcome	A comparison of the probabilities of rolling a single regular dice with the probabilities of rolling two dice simultaneously can emphasise the difference between uniform and non-uniform discrete random variables.
How can estimates of probabilities be obtained for discrete random variables from relative frequencies and probability bar charts? • When a large number of independent trials is considered, the relative frequency of an event gives an approximation for the probability of that event	Frequency tables and probability bar charts for simple discrete random variables (such as the number of times each student must roll a dice before getting a 6) are converted into probability distributions, which are then interpreted. Dice with 'alternative labelling' (such as 1, 1, 2, 3, 4, 5 on the six sides) are also used to create tables showing relative frequencies. Increasing the number of rolls of such dice shows how the approximation of the relative frequency approaches the theoretical probability. Electronic technology is used to simulate very large sets of data.
How can the expected value of a discrete random variable be calculated and used? • The expected value of a discrete random variable is calculated using $E(X)=\sum x.p(x)=\mu_X$, where $p(x)$ is the probability function for achieving result x and μ_X is the mean of the distribution • The principal purpose of the expected value is to be a measurement of the centre of the distribution • The expected value can be interpreted as a long-run sample mean	Probability distributions for familiar scenarios, such as the number of burgers purchased by people who enter a takeaway outlet, are used to show the usefulness of the expected value. The expected value for the previously described scenario of the tradesperson quoting $1000 for a job would give the tradesperson guidance on whether the quotes are too high or too low. An analogy for the expected value is the balancing point: if the probability bar chart is considered as stacked weights on a movable pivot then the system balances when the pivot is placed at the position of the expected value.  It is appropriate at this point to introduce discrete random variables that do not have a real-world context, but are given in mathematical form, such as:

x	2	5	8
$\Pr(X=x)$	0.40	0.25	0.35

Key questions and key concepts

How is the standard deviation of a discrete random variable calculated and used?

- The standard deviation of a discrete random variable is calculated $\sigma_X = \sqrt{\sum[x-\mu_X]^2 p(x)}$ where μ_X is the expected value and $p(x)$ is the probability function for achieving result x
- The principal purpose of the standard deviation is to be a measurement of the spread of the distribution

Considerations for developing teaching and learning strategies

The spreads of discrete random variables are compared using the standard deviation.

Subtopic 2.2: The Bernoulli distribution

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is a Bernoulli random variable, and where are they seen? • Discrete random variables with only two outcomes are called Bernoulli random variables. These two outcomes are often labelled 'success' and 'failure' • The Bernoulli distribution is the possible values and their probabilities of a Bernoulli random variable • One parameter, p, the probability of 'success', is used to describe Bernoulli distributions	The probability distribution of different Bernoulli random variables can be introduced by considering scenarios such as tossing a coin or rolling a dice that has different numbers of sides painted with one of two colours. These probabilities can be shown in a table or on a probability bar chart. The formula $p(x)=p^x(1-p)^{1-x}$, for $x=1$ signifying success and $x=0$ signifying failure is used for Bernoulli distributions. Other examples of Bernoulli distributions include whether or not an email is read within an hour of being sent, and the random selection of a tool and discovering whether it is defective or not.
What is the mean and standard deviation of the Bernoulli distribution? • The mean of the Bernoulli distribution is p, and the standard deviation is given by $\sqrt{p(1-p)}$	Investigation of the means of different Bernoulli distributions, calculated using $$E(X)=\sum x.p(x),$$ leads students to conjecture that the mean of a Bernoulli distribution is equal to p. This result can be proved using the algebraic formula for Bernoulli distributions. A similar investigation for the standard deviation may be used with guidance for formulating the conjecture, followed by completion of the algebraic proof. In this subject, the principal purpose of these values is to determine the mean and standard deviation for the binomial distribution.

Subtopic 2.3: Repeated Bernoulli trials and the binomial distribution

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is a binomial distribution and how is it related to Bernoulli trials? • When a Bernoulli trial is repeated, the number of successes is classed as a binomial random variable • The possible values for the different numbers of successes and their probabilities make up a binomial distribution	Students use tree diagrams to build simple binomial distributions such as the number of heads resulting when a coin is tossed three times.
What is the mean and standard deviation of the binomial distribution? • The mean of the binomial distribution is np, and the standard deviation is given by $\sqrt{np(1-p)}$, where p is the probability of success in a Bernoulli trial and n is the number of trials	Both of these expressions can be built by considering a binomial variable as the sum of n Bernoulli variables. The mean and standard deviation of the Bernoulli distribution are used to obtain these expressions. In this subject, the principal purpose of these values is to determine the mean and standard deviation for the distribution of sample proportions in Topic 6: Sampling and confidence intervals.
When can a situation be modelled using the binomial distribution? • A binomial distribution is suitable when the number of trials is fixed in advance, the trials are independent, and each trial has the same probability of success	Students classify scenarios as ones that can or cannot be modelled with a binomial distribution. In a context where sampling is done without replacement (e.g. an opinion poll), the distribution is not strictly binomial. When the population is large in comparison with the sample, the binomial distribution provides an excellent approximation of the probability of success.
How can binomial probabilities be calculated? • The probability of k successes from n trials is given by $\Pr(X=k)=C_k^n p^k (1-p)^{n-k}$, where p is the probability of success in the single Bernoulli trial	The tree diagram used to determine the probability of obtaining a 6 on a dice twice when the dice is rolled three times, is considered to show that the calculation of binomial probabilities involves: • calculation of the number of ways that k successes can occur within n events (using the concepts in Topic 4: Counting and statistics in Stage 1 Mathematics) • probability of success to the power of the required number of successes • probability of failure to the power of the required number of failures.

Key questions and key concepts	Considerations for developing teaching and learning strategies
How can electronic technology be used to calculate binomial probabilities?	Although an understanding of the binomial probability formula is a requirement of this subject, calculations of binomial probabilities in problem-solving should be done using electronic technology. Students, given the probability of success, calculate probabilities such as: • exactly k successes out of n trials • at least k successes out of n trials • between k_1 and k_2 successes out of n trials
What happens to the binomial distribution as n gets larger and larger? • The binomial distribution for large values of n has a symmetrical shape that many students will recognise	Students will not have been formally introduced to the normal distribution yet, but it may be familiar to many. Probability bar charts for binomial distributions of $n=10,\ n=100,\ n=1000$ are built using a spreadsheet (for a specific value of p). Comparing these graphs shows the shape approaching a normal distribution. Discussion on adding a smooth curve to emphasise the shape leads to the concept of continuous random variables.

Topic 3: Integral calculus

Subtopic 3.1: Anti-differentiation

Key questions and key concepts	Considerations for developing teaching and learning strategies
Is there an operation which is the reverse of differentiation? • Finding a function whose derivative is the given function is called 'anti-differentiation' • Anti-differentiation is more commonly called 'integration' or 'finding the indefinite integral'	Briefly consider simple derivatives. Posing a question about determining the original function, if its derivative is known, introduces integral calculus.
What is the indefinite integral $\int f(x)\,\mathrm{d}x$? • Any function $F(x)$ such that $F'(x)=f(x)$ is called the indefinite integral of $f(x)$ • The notation used for determining the indefinite integral is $\int f(x)\,\mathrm{d}x$	Introduce the formal language and notation of integration before exploring the different processes in detail.
Are there families of curves that have the same derivatives? • All families of functions of the form $F(x)+c$ for any constant c have the same derivative. Hence, if $F(x)$ is an indefinite integral of $f(x)$, then so is $F(x)+c$ for any constant c	
What are some types of functions that can be integrated? • By reversing the differentiation processes, the integrals of x^n (for $n\neq -1$), e^x, $\sin x$, and $\cos x$ can be determined	The processes for evaluating these integrals are developed by considering the relevant differentiation processes, and examining what must be done to reverse them.
• Reversing the differentiation processes and consideration of the chain rule can be used to determine the integrals of $\left[f(x)\right]^n$ (for $n\neq -1$), $e^{f(x)}$, $\sin f(x)$, and $\cos f(x)$ for linear functions $f(x)$ • $\int\left[f(x)+g(x)\right]\mathrm{d}x=\int f(x)\,\mathrm{d}x+\int g(x)\,\mathrm{d}x$	The use of substitution to find the integral of nested functions is not required.

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is needed to be able to determine a specific constant of integration? • When the value of the indefinite integral is known for a specific value of the variable (often an initial condition), the constant of integration can be determined • Determine the displacement given the velocity in a linear motion problem	Determine $f(x)$, given $f'(x)$ and $f(a)=b$.

Subtopic 3.2: The area under curves

Key questions and key concepts	Considerations for developing teaching and learning strategies
How can the area under a curve be estimated? • The area under a simple positive monotonic curve is approximated by upper and lower sums of the areas of rectangles of equal width	This subtopic can be introduced by briefly considering the areas of polygons, then discussing the difficulty of generalising this concept to figures with curved boundaries. Students start with simple graphs where area is worked out using geometric principles (e.g. cross-sectional areas, and the determination of approximation of π or other irrational numbers).
How can the estimate of the area be improved? • Decreasing the width of the rectangles improves the estimate of the area, but makes it more cumbersome to calculate	After students have calculated areas under a curve using electronic technology, they consider what process the technology is using to evaluate these areas by addition and multiplication operations only.
How can integration be used to find the exact value of the area under a curve? • The exact value of the area is the unique number between the upper and lower sums, which is obtained as the width of the rectangles approaches zero • The definite integral $\int_a^b f(x)\,\mathrm{d}x$ can be interpreted as the exact area of the region between the curve $y=f(x)$ and the x-axis over the interval $a \le x \le b$ (for a positive continuous function $f(x)$)	Students are introduced to the correct notation and use electronic technology to calculate areas. In many cases the technology will only give an approximation for the area, leading to the need for an algebraic process to calculate the exact value of definite integrals. This subtopic can be developed without reference to the algebraic computation of the antiderivative.
How is the area above a function that is below the x-axis calculated? • When $f(x)$ is a continuous negative function the exact area of the region between the curve $y=f(x)$ and the x-axis over the interval $a \le x \le b$ is given by: $-\int_a^b f(x)\,\mathrm{d}x$	The computation of carefully chosen integrals (using electronic technology) leads students to observe the following results: • $\int_a^a f(x)\,\mathrm{d}x = 0$ • $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$
How is the area between the functions $f(x)$ and $g(x)$ over the interval $a \le x \le b$ calculated? • When $f(x)$ is above $g(x)$, that is $f(x) \ge g(x)$, the area is given by $\int_a^b \left[f(x) - g(x)\right]\mathrm{d}x$	

Subtopic 3.3: Fundamental theorem of calculus

Key questions and key concepts	Considerations for developing teaching and learning strategies
How can exact values of definite integrals be calculated? • The statement of the fundamental theorem of calculus $$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$ where $F(x)$ is such that $F'(x) = f(x)$	The fundamental theorem of calculus is now introduced to address the need for exact values for areas under curves.
What results can be interpreted from the direct application of the fundamental theorem of calculus? • $\int_a^a f(x)\,\mathrm{d}x = 0$ • $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$	These two results may have been discovered when the areas under curves were being explored. They can now be verified by applying the fundamental theorem of calculus. The theorem $F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$ can be explored, and proved geometrically to show that the fundamental theorem of calculus links the processes of differentiation and integration.
How can the fundamental theorem of calculus be applied to evaluate areas? • In the exploration of areas in Subtopic 3.2, the use of technology meant that the exact value of areas (or the exact value of one of the end points of the area) could not always be obtained. The fundamental theorem of calculus can be used in those circumstances	

Subtopic 3.4: Applications of integration

Key questions and key concepts	Considerations for developing teaching and learning strategies
What useful information can be obtained by calculating the area under a curve? • Applications can be modelled by functions, and evaluating the area under or between curves can be used to solve problems • When the rate of change of a quantity is graphed against the elapsed time, the area under the curve is the total change in the quantity	The types of functions that may be suitable for this subtopic are given in Subtopic 3.1. When curves describe, for example, caves, tunnels, and drainage pipe cross-sections, the calculation of different areas has practical applications.
How can integration be used in motion problems? • The total distance travelled by an object is determined from its velocity function • An object's position is determined from its velocity function if the initial position (or position at some specific time) is known • An object's velocity is determined from its acceleration function if the initial velocity (or velocity at some specific time) is known	The following examples of rates of change models can be considered: • The rate at which people enter a sports venue • Water flow during a storm • The velocity of a vehicle • Traffic flow through a city • The electricity consumption of a household.

Topic 4: Logarithmic functions

Subtopic 4.1: Using logarithms for solving exponential equations

Key questions and key concepts	Considerations for developing teaching and learning strategies
How can an exact solution to an equation be obtained where the power is the unknown quantity? • The solution for x of the exponential equation $a^x = b$ is given using logarithms $x = \log_a b$	Logarithms were introduced in Subtopic 5.3 in Topic 5: Growth and decay, Stage 1 Mathematics. Logarithms are reviewed here for their use in algebraically solving exponential equations. Briefly consider the definition of a logarithm as a number, and the rules for operating with logarithms. Although logarithms of base a are a suitable revision of content from Stage 1 Mathematics, the focus of this topic is the natural logarithm.
Given $y = e^x$ what value of x will produce a given value of y? • The natural logarithm is the logarithm of base e • When $y = e^x$ then $x = \log_e y = \ln y$ • Natural logarithms obey the laws: $\ln a^b = b \ln a$ $\ln ab = \ln a + \ln b$ $\ln \frac{a}{b} = \ln a - \ln b$	Examining a table of the non-negative integer powers of e enables students to understand the extent and rapidity of exponential growth, and also raises the following questions: • In a population growth model $P = P_0 e^{kt}$, when will the population reach a particular value? • In a purely mathematical sense, what power of e gives 2, or 5, and so on? Looking for some of these values (by trial and error with a calculator) gives approximations for some specific natural logarithms, encountered previously as the multipliers for the derivatives of 2^x and 5^x.
How are natural logarithms used to find the solutions for problems involving applications of differential calculus with exponential functions? Derivatives are used to: • find the slope of tangents to the graphs of logarithmic functions, and hence the equation of the tangent and/or normal at any point on the function • find the instantaneous velocity, when an object's displacement is described by an exponential function • determine the rate of change, the position of any local maxima or minima, and when the function is increasing or decreasing	This subtopic allows for extension of the problem-solving approaches used in exploring applications of differential calculus. The use of logarithms allows solutions to be found. An example of suitable functions to explore is the logistic function: $P(t) = \frac{L}{1 + Ae^{-bt}}$.

Subtopic 4.2: Logarithmic functions and their graphs

Key questions and key concepts	Considerations for developing teaching and learning strategies
Where and why are logarithmic scales used?	
• In many areas of measurement, a logarithmic scale is used to render an exponential scale linear or because the numbers cover too large a range to make them easy to use	Examples of suitable logarithmic scales include magnitude of earthquakes (Richter scale), loudness (decibel scale), acidity (pH scale), and brightness of stars (Pogson's scale of apparent magnitude). The study of these scales is used to reinforce the nature of the logarithmic function.
What are the features and shape of the graph of $y=\ln x$ and its translations? • The graph of $y=\ln x$ is continuously increasing, with an x-intercept at $x=1$ and a vertical asymptote $x=0$ • The graph of $y=k\ln\left(b\left(x-c\right)\right)$ is the same shape as the graph of $y=\ln x$, with the values of k, b, and c determining its specific characteristics	Electronic technology can be used to graph $y=\log ax$ for different values of a, which enables students to formulate conjectures about the properties of this family of curves. Similarly, electronic technology can be used to explore the translations (caused by k, b, and c) within the graph of $y=k\ln\left(b\left(x-c\right)\right)$.
How are the graphs of $y=e^x$ and $y=\ln x$ related? • Like all inverse functions, the graphs of $y=e^x$ and $y=\ln x$ are reflections of each other in the line $y=x$	

Subtopic 4.3: Calculus of logarithmic functions

Key questions and key concepts	Considerations for developing teaching and learning strategies
What are the derivatives of $y=\ln x$ and $y=\ln f(x)$? • The function $y=\ln x$ has a derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{x}$ • The function $y=\ln f(x)$ has a derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{f'(x)}{f(x)}$	Using electronic technology to study the slope of tangents along the curve $y=\ln x$ leads students to the conjecture that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{x}$. The proof (using implicit differentiation) is not required in this subject. The derivative of $y=\ln f(x)$ can be established using the chain rule.
What is the indefinite integral of $\dfrac{1}{x}$? • Provided x is positive, $\int \dfrac{1}{x}\,\mathrm{d}x=\ln x+c$	The integral of $\dfrac{1}{x}$ can be considered by reviewing the relationships between functions and their derivatives through anti-differentiation.
What types of problems can be solved by finding the derivatives of logarithmic functions? The derivative of logarithmic functions can be used to: • find the slope of tangents to the function • find the instantaneous velocity, when an object's displacement is described by a logarithmic function • determine the rate of change, the position of any local maxima or minima, and when the function is increasing or decreasing	
Use logarithmic functions and their derivatives to solve practical problems.	Logarithmic functions can be used to model many scenarios, such as the magnitude of earthquakes, loudness, and acidity.

Topic 5: Continuous random variables and the normal distribution

Subtopic 5.1: Continuous random variables

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is a continuous random variable? • A continuous random variable can take any value (sometimes within set limits)	A comparison between discrete and continuous random variables can be made using people's ages (given in years) and heights.
How can the probabilities associated with continuous random variables be estimated? • The probability of each specific value of a continuous random variable is effectively zero. The probabilities associated with a specific range of values for a continuous random variable can be estimated from relative frequencies and from histograms	An exploration of histograms for decreasing width ranges of continuous random variables shows that a smooth curve would be suitable to illustrate probabilities.
How are probability density functions used? • A probability density function is a function that describes the relative likelihood for the continuous random variable to be a given value • A function is only suitable to be a probability density function if it is continuous and positive over the domain of the variable. Additionally, the area bound by the curve of the density function and the x-axis must equal 1, when calculated over the domain of the variable • The area under the probability density function from a to b gives the probability that the values of the continuous random variable are between a and b	The properties of all probability density functions can be explored using: • the uniform function $f\left(x\right)=\frac{1}{10}$ for $0\le x\le 10$ • the linear function $f\left(x\right)=\frac{x}{4}$ for $1\le x\le 3$ • the quadratic function $f\left(x\right)=\frac{3x}{2}-\frac{3x^2}{4}$ for $0\le x\le 2$. These functions would give the opportunity to revise the evaluation of areas using definite integrals (calculated using the fundamental theorem of calculus or using electronic technology). The fundamental theorem of calculus can be used for questions such as: • determine the value of k that makes $f\left(x\right)=ke^{-x}$ a probability density function in the domain $0\le x\le 1$.

Key questions and key concepts

How are the mean and standard deviation of continuous random variables calculated?

- the mean:

$$\mu_X = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x$$

- the standard deviation:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty} [x - \mu_X]^2 f(x)\,\mathrm{d}x}$$

Considerations for developing teaching and learning strategies

This section reinforces understanding of the usefulness of the mean as the expected value of a random variable — the long-run sample mean. Similarly, this section reinforces the role of the standard deviation in describing the spread of a random variable.

Approximations for definite integrals from $-\infty$ to ∞ can be obtained using technology to calculate $\int_{-A}^{A} x\, f(x)\mathrm{d}x$ with suitably large values of A.

Subtopic 5.2: Normal distributions

Key questions and key concepts	Considerations for developing teaching and learning strategies
What are normal random variables, and for what scenarios do they provide a suitable model? • Natural variation means that many random variables have the majority of values close to the mean, with symmetrically lower probabilities as the values get further above or below the mean. Under some conditions these variables can be classed as 'normal random variables'	When investigating an explanation for why normal distributions occur, students build a spreadsheet that generates a large amount of data from the sum of a finite number of random numbers. Potentially useful examples are found in manufacturing processes (e.g. the length of a concrete sleeper or the weight of a container of butter).
What are the key properties of normal distributions? • The normal distribution is symmetric and bell-shaped. Each normal distribution is determined by the mean μ and the standard deviation σ	Investigations of different normal distributions demonstrate the symmetrical shape, the position of the mean, and the standard deviation's role in determining the spread. The importance of the standard deviation is emphasised using the $68:95:99.7\%$ rule.
What is the probability density function for the normal distribution? • $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	With the aid of electronic technology, students explore the features of the graph of the function $f(x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$ to confirm that it is a probability density function. Different values of μ and σ are used to explore the effects of these parameters on transforming the density function for different distributions. The position of the inflections, in terms of standard deviations, is explored using the skills learned in the calculus topics.
How can the percentage of a population meeting a certain criterion be calculated for normal distributions? • When the normal distribution is used to model data, the area under the curve within an interval is interpreted as a proportion or a probability	The normal distribution functions of electronic technology should be used, rather than evaluating areas using the probability density function for calculations of proportions or probabilities.
How can the normal distribution be used to determine the value above or below which a certain proportion lies? • When one limit of a known area (upper or lower) is known, the other can be obtained	Electronic technology should be used to calculate, within a normal distribution, the upper or lower bounds of an interval that contains a specific proportion of the population.

Key questions and key concepts

What is the standard normal distribution?

- All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

Considerations for developing teaching and learning strategies

Through calculations, students develop an understanding that:

- for all normal curves, the value of $\Pr(\mu - a\sigma \le X \le \mu + b\sigma)$ does not depend on μ and σ, and is equal to $\Pr(a \le Z \le b)$.

Z can be interpreted as the number of standard deviations by which X lies above or below the mean.

The standard normal distribution provides one basis for the derivation of confidence intervals in Topic 6: Sampling and confidence intervals.

Subtopic 5.3: Sampling

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is the result when independent observations of a random variable X, with mean μ and standard deviation σ, are added together? • For n independent observations of X, the sum of the observations $\left(X_1+X_2+X_3+\ldots+X_n\right)$ is a random variable S_n • The distribution of S_n is called its sampling distribution • The sampling distribution of S_n has mean $n\mu$ and standard deviation $\sigma\sqrt{n}$	Students can start by collecting observations of random variables, for example, by rolling dice, and progress to using electronic technology to simulate random processes. One collection of n observations is called a sample. Once students understand the ideas behind the simulation process, they can take many samples from a distribution, calculate their sum, and analyse the distribution of these values. A variety of distributions should be used. The sizes of samples taken should be varied so that the characteristics of sampling distributions can be deduced. The distribution of S_n can be calculated explicitly in simple cases, for example, if a fair dice is rolled twice and the values of the two faces added. A binomial variable X can be considered as the sum of n independent Bernoulli variables. This leads to the fact that X has mean np and standard deviation $\sqrt{np\left(1-p\right)}$.
What is the result when the sample mean of independent observations of a random variable X, with mean μ and standard deviation σ, is calculated? • For n independent observations of X, the sample mean of the observations $\left(\dfrac{X_1+X_2+X_3+\ldots+X_n}{n}\right)$ is a random variable $\overline{X}_n$ • The distribution of $\overline{X}_n$ is called its sampling distribution • The distribution of the random variable $\overline{X}_n$ has a mean μ and standard deviation $\dfrac{\sigma}{\sqrt{n}}$	The sample mean of independent random variables is related simply to the sum of the variables by the factor $\dfrac{1}{n}$. All conclusions about the sampling distribution of the sum correspond to conclusions about the sampling distribution of the sample mean.
What is the form of the sampling distributions of S_n and $\overline{X}_n$ if X is normally distributed? • For any value of n, S_n is normally distributed, with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$	Simulated observations from a normal random variable can be obtained using electronic technology and used to illustrate that the sum and the sample mean are always normally distributed.

Key questions and key concepts	Considerations for developing teaching and learning strategies
• For any value of n, $\overline{X}_n$ is normally distributed, with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ What is the form of the sampling distributions of S_n and $\overline{X}_n$ if the distribution of X is not normally distributed? • Provided n is sufficiently large, S_n is approximately normally distributed, with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$ • Provided n is sufficiently large, $\overline{X}_n$ is approximately normally distributed, with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ This result is commonly known as the central limit theorem.	Students can simulate observations from non-normal random variables. Many samples from a non-normally distributed random variable can be taken, their sum or sample mean calculated, and the distribution of these values analysed. The sizes of samples taken should be varied so that students can appreciate that the size of n that provides approximate normality depends on the degree of non-normality of the population distribution. Students could explore problems such as calculating the probability of buying an individual can of soft drink (labelled 375 mL) that contains less than 375 mL compared with the probability of buying a six-pack of the same cans of soft drink with an average content of less than 375 mL. The normal approximation to the binomial distribution can be seen as an example of the central limit theorem.
What is a simple random sample of observations from a population? • A simple random sample of size n is a collection of n subjects chosen from a population in such a way that every possible sample of size n has an equal chance of being selected	A simple random sample can be selected from a finite population by sampling randomly without replacement.
How is a simple random sample linked to sampling distributions? • If one simple random sample of n individuals is chosen from a population, and the value of a certain variable X is recorded for each individual in the sample, the sample mean of the n values for this one sample $\overline{x}$ is one observation of the random variable denoted $\overline{X}_n$ • If X has population mean μ and population standard deviation σ, then the sampling distribution of $\overline{X}_n$ is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ provided that the sample size n is sufficiently large but also small compared with the population size N • Approximate probabilities for $\overline{X}_n$ can be calculated from the sampling distribution	When a single subject is sampled from the population and the value of X is recorded, the distribution of X is a discrete distribution that corresponds to the population histogram. For sampling with replacement from a finite population, the results for the sum and sample mean of independent random variables apply directly. For a simple random sample, the results for the sum and sample mean of independent random variables apply approximately, provided the sample size is small relative to the population size. This is because sampling with and without replacement behave similarly when the sample size is small relative to the population size. Students can conduct physical and electronic sampling experiments to illustrate the properties of sample means of simple random samples and explore the effects of sample size and population composition.

Topic 6: Sampling and confidence intervals

Subtopic 6.1: Confidence intervals for a population mean

Key questions and key concepts	Considerations for developing teaching and learning strategies
What type of random variable is a sample mean? • If a sample is chosen and its mean calculated then the value of that sample mean will be variable. Different samples will yield different sample means • Sample means are continuous random variables • For a sufficiently large sample size, the distribution of sample means will be approximately normal. The distribution of sample means will have a mean equal to μ, the population mean. This distribution has a standard deviation equal to $\frac{\sigma}{\sqrt{n}}$, where σ is the standard deviation of the population and n is the sample size	Students are familiar with the concept of sample means from study of Subtopic 5.3: Sampling. Briefly consider that a sample mean cannot be used to make definitive statements about the population mean, but can be used as a guide only. Students explored the distribution of sample means in Subtopic 5.3: Sampling.
How can a single sample mean (of appropriately large sample size) be used to create an interval estimate for the population mean? • An interval can be created around the sample mean that will be expected, with some specific confidence level, to contain the population mean	Knowing the approximately normal distribution of sample means and the $68:95:99.7$% rule, it can be seen that approximately 68% of sample means will be within one standard deviation of the unknown population mean and that approximately 95% of sample means will be within two standard deviations of the unknown population mean.
How can the upper and lower limits of a confidence interval for the population mean be calculated? • If $\overline{x}$ is the sample mean and s the standard deviation of a suitably large sample, then the interval: $\overline{x}-z\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x}+z\frac{s}{\sqrt{n}}$ can be created. The value of z is determined by the confidence level required	This formula can be introduced by considering the approximate 68% confidence interval using $z=1$ and the approximate 95% confidence interval using $z=2$. The creation of both of these intervals shows students that σ, not s, should be used if available, but rarely is σ known when μ is not. Other confidence levels (say 90% and 98%) require the calculation of z using students' knowledge of the standard normal distribution and of how to calculate (using electronic technology) the upper or lower bounds of an interval within a normal distribution that contains a specific proportion of the population. Although an understanding of the confidence interval formula is a requirement of this subject, calculations of confidence intervals in problem-solving should be done using electronic technology.

Key questions and key concepts	Considerations for developing teaching and learning strategies
Can a confidence interval be used to state facts about a population mean? • Not all confidence intervals will contain the true population mean • The inclusion or not of a claimed population mean within a confidence interval can be used a guide to whether the claim is true or false, but definitive statements are not possible	Simulation (using electronic technology to calculate the different intervals) can show the variation in confidence intervals (especially their approximate margin of error $z\frac{s}{\sqrt{n}}$) caused by different sample means, different sample standard deviations, and different confidence levels.

Subtopic 6.2: Population proportions

Key questions and key concepts	Considerations for developing teaching and learning strategies
What is a population proportion and what does it represent? • A population proportion p is the proportion of elements in a population that have a given characteristic. It is usually given as a decimal or fraction • A population proportion represents the probability that one element of the population, chosen at random, has the given characteristic being studied What is a sample proportion? • If a sample of size n is chosen, and X is the number of elements with a given characteristic, then the sample proportion $\hat{p}$ is equal to $\frac{X}{n}$	Discussions about the number of left-handed students in a class can introduce the concept of a sample proportion. Compare this to the proportion of a population that is left-handed. By quickly surveying nearby classes, the effect of sample size on how close a sample proportion is to a known population proportion can be explored.
What type of variable is a sample proportion? • A sample proportion is a discrete random variable. The distribution has a mean of p and a standard deviation of $\sqrt{\frac{p(1-p)}{n}}$ What happens to the distribution of $\hat{p}$ for large samples? • As the sample size increases, the distribution of $\hat{p}$ becomes more and more like a normal distribution	To justify that a sample proportion is discrete, students consider a sample size of 10. Sample proportions of $\frac{5}{10}$ (0.5) and $\frac{6}{10}$ (0.6) would be possible, but a sample proportion of 0.55 would not. The mean and standard deviation can easily be derived from the mean and standard deviation of the binomial distribution.

Subtopic 6.3: Confidence intervals for a population proportion

Key questions and key concepts	Considerations for developing teaching and learning strategies
Can facts about a population proportion be gained from a sample proportion? • An interval can be created around the sample proportion that will be expected, with some specific confidence level, to contain the population proportion	
How can the upper and lower limits of a confidence interval for the population proportion be calculated? • If $\hat{p}$ is the sample proportion, then the interval $\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ can be created. The value of z is determined by the confidence level required	This formula matches the structure of the formula for the confidence interval for a population mean, and allows for revision of its creation by considering the position within the distributions (compared to the population proportion) of individual samples proportions. Although an understanding of the confidence interval formula is a requirement of this subject, calculations of confidence intervals in problem-solving should be done using electronic technology.
Can a confidence interval be used to state facts about a population proportion? • Not all confidence intervals will contain the true population proportion • The inclusion or not of a claimed population proportion within a confidence interval can be used a guide to whether the claim is true or false, but definitive statements are not possible	Simulation (using electronic technology to calculate the different intervals) can show the variation in confidence intervals (especially their approximate margin of error $z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$) caused by different sample proportions and different confidence levels.

ASSESSMENT SCOPE AND REQUIREMENTS

All Stage 2 subjects have a school assessment component and an external assessment component.

EVIDENCE OF LEARNING

The following assessment types enable students to demonstrate their learning in Stage 2 Mathematical Methods.

School assessment (70%)

- Assessment Type 1: Skills and Applications Tasks (50%)
- Assessment Type 2: Mathematical Investigation (20%)

External assessment (30%)

- Assessment Type 3: Examination (30%)

Students provide evidence of their learning through eight assessments, including the external assessment component. Students undertake:

- six skills and applications tasks
- one mathematical investigation
- one examination.

ASSESSMENT DESIGN CRITERIA

The assessment design criteria are based on the learning requirements and are used by:

- teachers to clarify for students what they need to learn
- teachers and assessors to design opportunities for students to provide evidence of their learning at the highest possible level of achievement.

The assessment design criteria consist of specific features that:

- students should demonstrate in their learning
- teachers and assessors look for as evidence that students have met the learning requirements.

For this subject, the assessment design criteria are:

- concepts and techniques
- reasoning and communication.

The specific features of these criteria are described below.

The set of assessments, as a whole, must give students opportunities to demonstrate each of the specific features by the completion of study of the subject.

Concepts and Techniques

The specific features are as follows:

CT1 Knowledge and understanding of concepts and relationships.

CT2 Selection and application of mathematical techniques and algorithms to find solutions to problems in a variety of contexts.

CT3 Application of mathematical models.

CT4 Use of electronic technology to find solutions to mathematical problems.

Reasoning and Communication

The specific features are as follows:

RC1 Interpretation of mathematical results.

RC2 Drawing conclusions from mathematical results, with an understanding of their reasonableness and limitations.

RC3 Use of appropriate mathematical notation, representations, and terminology.

RC4 Communication of mathematical ideas and reasoning to develop logical arguments.

RC5 Development, testing, and proof of valid conjectures.*

* In this subject students must be given the opportunity to develop, test, and prove conjectures in at least one assessment task in the school assessment component.

SCHOOL ASSESSMENT

Assessment Type 1: Skills and Applications Tasks (50%)

Students complete six skills and applications tasks.

Skills and applications tasks are completed under the direct supervision of the teacher.

The equivalent of one skills and applications task must be undertaken without the use of either a calculator or notes.

In the remaining skills and applications tasks, electronic technology and up to one A4 sheet of paper of handwritten notes (on one side only) may be used at the discretion of the teacher.

Students find solutions to mathematical problems that may:

- be routine, analytical, and/or interpretative
- be posed in a variety of familiar and new contexts
- require discerning use of electronic technology.

In setting skills and applications tasks, teachers may provide students with information in written form or in the form of numerical data, diagrams, tables, or graphs. A task should require students to demonstrate an understanding of relevant mathematical concepts and relationships.

Students select appropriate techniques or algorithms and relevant mathematical information to find solutions to routine, analytical, and/or interpretative problems. Some of these problems should be set in context, for example, social, scientific, economic, or historical.

Students provide explanations and arguments, and use correct mathematical notation, terminology, and representations throughout the task.

Skills and applications tasks may provide opportunities to develop, test, and prove conjectures.

For this assessment type, students provide evidence of their learning in relation to the following assessment design criteria:

- concepts and techniques
- reasoning and communication.

Assessment Type 2: Mathematical Investigation (20%)

Students complete one mathematical investigation.

Students investigate mathematical relationships, concepts, or problems, which may be set in an applied context. The subject of a mathematical investigation may be derived from one or more subtopics, although it can also relate to a whole topic or across topics.

A mathematical investigation may be initiated by a student, a group of students, or the teacher. Teachers should give some direction about the appropriateness of each student’s choice, and guide and support students’ progress in an investigation. For this investigation there must be minimal teacher direction and teachers must allow the opportunity for students to extend the investigation in an open-ended context.

Students demonstrate their problem-solving strategies as well as their knowledge, skills, and understanding in the investigation. They are encouraged to use a variety of mathematical and other software (e.g. computer algebra systems (CAS), spreadsheets, statistical packages) to enhance their investigation. The generation of data and the exploration of patterns and structures, or changing parameters, may provide an important focus. From these, students may recognise different patterns or structures. Notation, terminology, forms of representation of information gathered or produced, calculations, evidence of technological skills, and results are important considerations.

Students complete a report for the mathematical investigation.

In the report, students interpret and justify results, and draw conclusions. They give appropriate explanations and arguments. The mathematical investigation may provide an opportunity to develop, test, and prove conjectures.

The report may take a variety of forms, but would usually include the following:

- an outline of the problem and context
- the method required to find a solution, in terms of the mathematical model or strategy used
- the application of the mathematical model or strategy, including
  - relevant data and/or information
  - mathematical calculations and results, using appropriate representations
  - the analysis and interpretation of results, including consideration of the reasonableness and limitations of the results
- the results and conclusions in the context of the problem.

A bibliography and appendices, as appropriate, may be used.

The format of the investigation report may be written or multimodal.

The investigation report, excluding bibliography and appendices if used, must be a maximum of 15 A4 pages if written, or the equivalent in multimodal form. The maximum page limit is for single-sided A4 pages with minimum font size 10. Page reduction, such as two A4 pages reduced to fit on one A4 page, is not acceptable. Conclusions, interpretations and/or arguments that are required for the assessment must be presented in the report, and not in an appendix. Appendices are used only to support the report, and do not form part of the assessment decision.

For this assessment type, students provide evidence of their learning in relation to the following assessment design criteria:

- concepts and techniques
- reasoning and communication.

EXTERNAL ASSESSMENT

Assessment Type 3: Examination (30%)

Students undertake a 3-hour external examination.

The examination is based on the key questions and key concepts in the six topics. The considerations for developing teaching and learning strategies are provided as a guide only, although applications described under this heading may provide contexts for examination questions.

The examination consists of a range of problems, some focusing on knowledge and routine skills and applications, and others focusing on analysis and interpretation. Some problems may require students to interrelate their knowledge, skills, and understanding from more than one topic. Students provide explanations and arguments, and use correct mathematical notation, terminology, and representations throughout the examination.

A formula sheet is included in the examination booklet.

Students may take two unfolded A4 sheets (four sides) of handwritten notes into the examination room.

Students may have access to approved electronic technology during the external examination. However, students need to be discerning in their use of electronic technology to find solutions to questions/problems in examinations.

All specific features of the assessment design criteria for this subject may be assessed in the external examination.

PERFORMANCE STANDARDS

The performance standards describe five levels of achievement, A to E.

Each level of achievement describes the knowledge, skills, and understanding that teachers and assessors refer to in deciding how well students have demonstrated their learning, on the basis of the evidence provided.

During the teaching and learning program the teacher gives students feedback on their learning, with reference to the performance standards.

At the student's completion of study of each school assessment type, the teacher makes a decision about the quality of the student's learning by:

- referring to the performance standards
- assigning a grade between A+ and E– for the assessment type.

The student's school assessment and external assessment are combined for a final result, which is reported as a grade between A+ and E–.

Performance Standards for Stage 2 Mathematical Methods

	Concepts and Techniques	Reasoning and Communication
A	Comprehensive knowledge and understanding of concepts and relationships. Highly effective selection and application of mathematical techniques and algorithms to find efficient and accurate solutions to routine and complex problems in a variety of contexts. Successful development and application of mathematical models to find concise and accurate solutions. Appropriate and effective use of electronic technology to find accurate solutions to routine and complex problems.	Comprehensive interpretation of mathematical results in the context of the problem. Drawing logical conclusions from mathematical results, with a comprehensive understanding of their reasonableness and limitations. Proficient and accurate use of appropriate mathematical notation, representations, and terminology. Highly effective communication of mathematical ideas and reasoning to develop logical and concise arguments. Effective development and testing of valid conjectures, with proof.
B	Some depth of knowledge and understanding of concepts and relationships. Mostly effective selection and application of mathematical techniques and algorithms to find mostly accurate solutions to routine and some complex problems in a variety of contexts. Some development and successful application of mathematical models to find mostly accurate solutions. Mostly appropriate and effective use of electronic technology to find mostly accurate solutions to routine and some complex problems.	Mostly appropriate interpretation of mathematical results in the context of the problem. Drawing mostly logical conclusions from mathematical results, with some depth of understanding of their reasonableness and limitations. Mostly accurate use of appropriate mathematical notation, representations, and terminology. Mostly effective communication of mathematical ideas and reasoning to develop mostly logical arguments. Mostly effective development and testing of valid conjectures, with substantial attempt at proof.
C	Generally competent knowledge and understanding of concepts and relationships. Generally effective selection and application of mathematical techniques and algorithms to find mostly accurate solutions to routine problems in a variety of contexts. Successful application of mathematical models to find generally accurate solutions. Generally appropriate and effective use of electronic technology to find mostly accurate solutions to routine problems.	Generally appropriate interpretation of mathematical results in the context of the problem. Drawing some logical conclusions from mathematical results, with some understanding of their reasonableness and limitations. Generally appropriate use of mathematical notation, representations, and terminology, with reasonable accuracy. Generally effective communication of mathematical ideas and reasoning to develop some logical arguments. Development and testing of generally valid conjectures, with some attempt at proof.

	Concepts and Techniques	Reasoning and Communication
D	Basic knowledge and some understanding of concepts and relationships. Some selection and application of mathematical techniques and algorithms to find some accurate solutions to routine problems in some contexts. Some application of mathematical models to find some accurate or partially accurate solutions. Some appropriate use of electronic technology to find some accurate solutions to routine problems.	Some interpretation of mathematical results. Drawing some conclusions from mathematical results, with some awareness of their reasonableness or limitations. Some appropriate use of mathematical notation, representations, and terminology, with some accuracy. Some communication of mathematical ideas, with attempted reasoning and/or arguments. Attempted development or testing of a reasonable conjecture.
E	Limited knowledge or understanding of concepts and relationships. Attempted selection and limited application of mathematical techniques or algorithms, with limited accuracy in solving routine problems. Attempted application of mathematical models, with limited accuracy. Attempted use of electronic technology, with limited accuracy in solving routine problems.	Limited interpretation of mathematical results. Limited understanding of the meaning of mathematical results, their reasonableness or limitations. Limited use of appropriate mathematical notation, representations, or terminology, with limited accuracy. Attempted communication of mathematical ideas, with limited reasoning. Limited attempt to develop or test a conjecture.

ASSESSMENT INTEGRITY

The SACE Assuring Assessment Integrity Policy outlines the principles and processes that teachers and assessors follow to assure the integrity of student assessments. This policy is available on the SACE website (www.sace.sa.edu.au) as part of the SACE Policy Framework.

The SACE Board uses a range of quality assurance processes so that the grades awarded for student achievement, in both the school assessment and the external assessment, are applied consistently and fairly against the performance standards for a subject, and are comparable across all schools.

Information and guidelines on quality assurance in assessment at Stage 2 are available on the SACE website (www.sace.sa.gov.au).

SUPPORT MATERIALS

SUBJECT-SPECIFIC ADVICE

Online support materials are provided for each subject and updated regularly on the SACE website (www.sace.sa.edu.au). Examples of support materials are sample learning and assessment plans, annotated assessment tasks, annotated student responses, and recommended resource materials.

ADVICE ON ETHICAL STUDY AND RESEARCH

Advice for students and teachers on ethical study and research practices is available in the guidelines on the ethical conduct of research in the SACE on the SACE website (www.sace.sa.edu.au).